

Another convex combination of product states for the separable Werner state

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Abstract

In this paper, we write down the separable Werner state in a two-qubit system explicitly as a convex combination of product states, which is different from the convex combination obtained by Wootters' method. The Werner state in a two-qubit system has a single real parameter and varies from inseparable state to separable state according to the value of its parameter. We derive a hidden variable model that is induced by our decomposed form for the separable Werner state. From our explicit form of the convex combination of product states, we understand the following: The critical point of the parameter for separability of the Werner state comes from positivity of local density operators of the qubits.

1 Introduction

The Einstein-Podolsky-Rosen paradox and Bell's pioneering works reveal that no hidden variable model can reproduce all predictions of quantum mechanics [1, 2, 3]. Thus, quantum correlation is essentially different from classical correlation. Motivation of quantum information theory, which many researchers have been eager to study for the last several decades, is to obtain a deep understanding of the quantum correlation.

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A bipartite quantum system is separable if its density matrix can be written as a convex combination of product states. A separable quantum system always admits the hidden variable interpretation. However, the converse is not necessarily true. Werner constructs a family of bipartite states, which are characterized by a single real parameter. He shows some inseparable states that belong to this family admit the hidden variable interpretation [4]. The states of this family are called the Werner states. Moreover, Popescu indicates that the inseparable Werner states admitting hidden variable models reveal nonlocal correlation under a sequence of measurements, where the second measurement depends on an output of the first measurement [5].

A criterion of separability for a two-qubit system is conjectured by Peres and established by Horodecki *et al.* [6, 7]. In a two-qubit system, the Werner state has a single real parameter and varies from inseparable state to separable state according to the value of its parameter. Using Peres-Horodecki's criterion, we can fix the critical point of the parameter between the separable and inseparable states.

The Werner state is finding wide application in the quantum information processing. It often appears as an intermediate during the quantum purification protocol [8, 9]. Thus, we can expect that the Werner state plays an important role in the process of local quantum operations and classical communications (LQCC). Various properties of the Werner state under LQCC is investigated by Hiroshima and Ishizaka [10].

As mentioned above, the Werner state has many interesting properties. However, we do not know why the Werner state in a two-qubit system changes from inseparable state to separable state suddenly at the critical point of its parameter. To examine the physical meaning of the critical point, we have to know an explicit form of a convex combination of product states for the separable Werner state. In this paper, we investigate a convex combination of product states for the separable Werner state, which is different from the convex combination obtained by Wootters' method [11]. (A convex combination of product states for a given separable density matrix is not unique generally.)

The decomposition obtained by Wootters' method is an ensemble of four pure states. By contrast, our decomposition is an integral of a product state with a probability distribution function over a continuous variable. Our decomposed form seems simpler than the decomposed form obtained by Wootters' method, and thus it may give us some insight. This is an advantage of our result. Looking at our decomposed form, we can understand that the critical point of the parameter for separability of the Werner state comes from positivity of local density operators of the qubits. Furthermore, our result produces a hidden variable model because the convex combination of product states always causes the hidden variable interpretation.

Here, we give a brief summary of Wootters' results in Ref. [11]. Wootters shows an explicit formula for the entanglement of formation of an arbitrary two-qubit system as a function of its density matrix. We can judge whether or not a given two-qubit density matrix is entangled by a value of its entanglement of formation. The density matrix is not entangled if its entanglement of formation is equal to zero, and the density matrix is entangled if its entanglement of formation is more than zero. Thus, we can use Wootters' formula instead of Peres-Horodecki's criterion.

Wootters also shows how to construct an entanglement-minimizing decomposition of an arbitrary two-qubit density matrix. In this decomposition, the density matrix is de-

scribed by a convex combination of pure states, and the average entanglement of the pure states is equal to the entanglement of formation. Thus, if we decompose a separable two-qubit density matrix according to Wootters' method, we obtain an ensemble of pure states, each of which has no entanglement. Hence, in general, Wootters' decomposition gives us a convex combination of product states for a separable density matrix explicitly. In Appendix A, we write down the decomposition obtained by Wootters' method for the separable Werner state.

In the rest of this section, we introduce the Werner state for a two-qubit system and examine its separability by Peres-Horodeckis' criterion. In Sec. 2, we investigate the relation between the separable Werner state and the hidden variable interpretation. In Sec. 3, we derive the explicit form of the convex combination of product states for the separable Werner state. In Sec. 4 we give a brief discussion. In Appendix A, we describe the decomposition of the separable Werner state obtained by Wootters' method.

The Werner state is given by the following density operator on a four-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ spanned by two qubits A and B:

$$W(q) = q|\Psi^-\rangle\langle\Psi^-| + \frac{1-q}{4}\mathbf{I}_{(4)}, \quad (1)$$

where $0 \leq q \leq 1$. $\mathbf{I}_{(4)}$ is the identity operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. $|\Psi^-\rangle$ is one of the Bell states that are maximally entangled on the two-qubit system and it is given by

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B). \quad (2)$$

$W(q)$ defined by Eq. (1) satisfies

$$W(q)^\dagger = W(q), \quad (3)$$

$$\text{Tr}W(q) = 1, \quad (4)$$

$$\langle\psi|W(q)|\psi\rangle \geq 0 \quad \forall|\psi\rangle. \quad (5)$$

Because of the above properties, we can regard $W(q)$ as a density operator.

We can judge whether $W(q)$ is separable or inseparable, that is, whether $W(q)$ is disentangled or entangled, from Peres-Horodeckis' criterion. According to Peres-Horodeckis' criterion, defining the partial transposition of $W(q)$ as $\tilde{W}(q)$, $W(q)$ is separable if all eigenvalues of $\tilde{W}(q)$ are non-negative, and $W(q)$ is inseparable if one of eigenvalues of $\tilde{W}(q)$ is negative. Let us examine eigenvalues of $\tilde{W}(q)$ below.

First of all, we give a matrix representation of $W(q)$ in a ket basis $\{|i\rangle_A|j\rangle_B : i, j \in \{0, 1\}\}$ as follows:

$$W(q) = \frac{1}{4} \begin{pmatrix} 1-q & 0 & 0 & 0 \\ 0 & 1+q & -2q & 0 \\ 0 & -2q & 1+q & 0 \\ 0 & 0 & 0 & 1-q \end{pmatrix}. \quad (6)$$

Thus, we obtain a matrix representation of $\tilde{W}(q)$ as follows:

$$\tilde{W}(q) = \frac{1}{4} \begin{pmatrix} 1-q & 0 & 0 & -2q \\ 0 & 1+q & 0 & 0 \\ 0 & 0 & 1+q & 0 \\ -2q & 0 & 0 & 1-q \end{pmatrix}. \quad (7)$$

In Eq. (7), the density operator is subjected to transposition on the Hilbert space \mathcal{H}_B spanned by the qubit B. By some calculation, we obtain three-fold degenerate eigenvalues, $(1 + q)/4$, and the last eigenvalue, $(1 - 3q)/4$, for $\tilde{W}(q)$. Hence, $W(q)$ is separable for $0 \leq q \leq 1/3$ and inseparable for $1/3 < q \leq 1$.

2 The separable Werner state and the hidden variable interpretation

From the discussion given in the previous section, we find that $W(q)$ is separable for $0 \leq q \leq 1/3$. The separable $W(q)$ has to be rewritten as the convex combination of product states and therefore it admits the hidden variable interpretation. In this section, we investigate relation between the separability of $W(q)$ and the hidden variable interpretation.

In general, we can rewrite the separable $W(q)$ in the form:

$$W(q) = \sum_{\lambda} p_{\lambda} (\rho_{A,\lambda} \otimes \rho_{B,\lambda}), \quad (8)$$

where $0 \leq p_{\lambda} \leq 1$, $\sum_{\lambda} p_{\lambda} = 1$, and $\rho_{A,\lambda}$ and $\rho_{B,\lambda}$ represent density operators of the qubits A and B, respectively. Here, we may regard the index λ in Eq. (8) as a continuous variable. Moreover, we can describe an arbitrary one-qubit density operator ρ as

$$\rho = \frac{1}{2}(\mathbf{I}_{(2)} + \mathbf{a} \cdot \boldsymbol{\sigma}), \quad (9)$$

where $\mathbf{I}_{(2)}$ represents the identity operator on a two-dimensional Hilbert space spanned by a single qubit, and \mathbf{a} represents an arbitrary three-dimensional real vector whose norm is equal to or less than unity. $\boldsymbol{\sigma}$ stands for a three-dimensional vector whose three components are Pauli matrices, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

From the above consideration, we can rewrite $W(q)$ defined in Eq. (8) as follows:

$$\begin{aligned} W(q) &= \int d\lambda p(\lambda) (\rho_A(\lambda) \otimes \rho_B(\lambda)) \\ &= \int d\lambda p(\lambda) \frac{1}{2}(\mathbf{I}_{(2)} + \mathbf{a}(\lambda) \cdot \boldsymbol{\sigma})_A \otimes \frac{1}{2}(\mathbf{I}_{(2)} + \mathbf{b}(\lambda) \cdot \boldsymbol{\sigma})_B, \end{aligned} \quad (10)$$

where

$$\int d\lambda p(\lambda) = 1, \quad (11)$$

and

$$|\mathbf{a}(\lambda)| \leq 1, \quad |\mathbf{b}(\lambda)| \leq 1. \quad (12)$$

Although we write λ as a single variable in Eq. (10), we can consider that λ stands for multiple variables. Moreover, because λ , $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$ depend on q , strictly speaking, we have to write them as $\lambda(q)$, $p(q, \lambda(q))$, $\mathbf{a}(q, \lambda(q))$, and $\mathbf{b}(q, \lambda(q))$. However, for simplicity, we omit q from their notations. [We never insist that λ , $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$ do not depend on q .] Furthermore, we have to pay attention to the fact that the convex combination of product states for $W(q)$ given in Eq. (10) is not unique.

The convex combination of product states for $W(q)$ given in Eq. (10) admits the hidden variable interpretation. We can understand this fact from the following explanation. Let us perform orthogonal measurements by Hermitian operators, $E(\mathbf{l})_A$ and $E(\mathbf{m})_B$, on the qubits A and B, respectively. We assume that $E(\mathbf{l})_A$ and $E(\mathbf{m})_B$ are given by the following form:

$$\begin{cases} E(\mathbf{l})_A = \mathbf{l} \cdot \boldsymbol{\sigma}_A & \text{for } |\mathbf{l}| = 1, \\ E(\mathbf{m})_B = \mathbf{m} \cdot \boldsymbol{\sigma}_B & \text{for } |\mathbf{m}| = 1. \end{cases} \quad (13)$$

An expectation value of the outcome in the measurement on the qubit A is given by

$$\frac{1}{2} \text{Tr}[E(\mathbf{l})(\mathbf{I}_{(2)} + \mathbf{a}(\lambda) \cdot \boldsymbol{\sigma})] = \frac{1}{2} \text{Tr}[(\mathbf{l} \cdot \boldsymbol{\sigma})(\mathbf{a} \cdot \boldsymbol{\sigma})] = \mathbf{l} \cdot \mathbf{a}. \quad (14)$$

Equation (14) implies that we obtain 1 as an output with probability $(1 + \mathbf{l} \cdot \mathbf{a})/2$ and we obtain (-1) as an output with probability $(1 - \mathbf{l} \cdot \mathbf{a})/2$ in the measurement on the qubit A. We obtain a similar result on the qubit B.

Therefore, we can describe an expectation value of a product of two outputs obtained from the measurements on the qubits A and B as follows:

$$\begin{aligned} C(\mathbf{l}, \mathbf{m}) &= \text{Tr}[(E(\mathbf{l})_A \otimes E(\mathbf{m})_B)W(q)] \\ &= \int d\lambda \int_0^1 d\lambda_A \int_0^1 d\lambda_B p(\lambda) A(\lambda, \lambda_A; \mathbf{l}) B(\lambda, \lambda_B; \mathbf{m}), \end{aligned} \quad (15)$$

where

$$A(\lambda, \lambda_A; \mathbf{l}) = \begin{cases} 1 & \text{for } 0 \leq \lambda_A \leq (1 + \mathbf{l} \cdot \mathbf{a})/2, \\ -1 & \text{for } (1 + \mathbf{l} \cdot \mathbf{a})/2 < \lambda_A \leq 1, \end{cases} \quad (16)$$

and

$$B(\lambda, \lambda_B; \mathbf{m}) = \begin{cases} 1 & \text{for } 0 \leq \lambda_B \leq (1 + \mathbf{m} \cdot \mathbf{b})/2, \\ -1 & \text{for } (1 + \mathbf{m} \cdot \mathbf{b})/2 < \lambda_B \leq 1. \end{cases} \quad (17)$$

This is a hidden variable model.

3 Decomposition of the separable Werner state

In this section, we derive $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$ given in Eq. (10) explicitly. First, we examine the expectation value of the output that is obtained by the measurement of $E(\mathbf{l})_A$ on the qubit A of $W(q)$. At first we calculate this expectation value from Eq. (1), and then we calculate it from Eq. (10). Next, we compare these two results.

From Eq. (1), we obtain

$$\text{Tr}[(E(\mathbf{l})_A \otimes \mathbf{I}_B)W(q)] = q \langle \Psi^- | E(\mathbf{l})_A \otimes \mathbf{I}_B | \Psi^- \rangle = 0. \quad (18)$$

On the other hand, from Eq. (10), we obtain

$$\text{Tr}[(E(\mathbf{l})_A \otimes \mathbf{I}_B)W(q)] = \int d\lambda p(\lambda) (\mathbf{l} \cdot \mathbf{a}(\lambda)). \quad (19)$$

Comparing Eqs. (18) and (19), we obtain the following relation:

$$\int d\lambda p(\lambda) (\mathbf{l} \cdot \mathbf{a}(\lambda)) = 0 \quad \forall |\mathbf{l}| \leq 1. \quad (20)$$

Thus, we arrive at

$$\int d\lambda p(\lambda) a_i(\lambda) = 0 \quad i \in \{x, y, z\}. \quad (21)$$

Examining the orthogonal measurement performed on the qubit B of $W(q)$, we can give a similar discussion and we obtain the following result:

$$\int d\lambda p(\lambda) b_i(\lambda) = 0 \quad i \in \{x, y, z\}. \quad (22)$$

Second, we examine the expectation value of the product of the outputs that we obtain by the measurements of $E(\mathbf{l})_A$ and $E(\mathbf{m})_B$ on the qubits A and B of $W(q)$, respectively. At first we calculate this expectation value from Eq. (1), and then we calculate it from Eq. (10). Next, we compare these two results. From Eq. (1), we obtain

$$\text{Tr}[(E(\mathbf{l})_A \otimes E(\mathbf{m})_B)W(q)] = q\langle\Psi^-|E(\mathbf{l})_A \otimes E(\mathbf{m})_B|\Psi^-\rangle = -q(\mathbf{l} \cdot \mathbf{m}). \quad (23)$$

On the other hand, from Eq. (10), we obtain

$$\text{Tr}[(E(\mathbf{l})_A \otimes E(\mathbf{m})_B)W(q)] = \int d\lambda p(\lambda)(\mathbf{l} \cdot \mathbf{a}(\lambda))(\mathbf{m} \cdot \mathbf{b}(\lambda)). \quad (24)$$

Comparing Eqs. (23) and (24), we obtain the following relation:

$$\int d\lambda p(\lambda)(\mathbf{l} \cdot \mathbf{a}(\lambda))(\mathbf{m} \cdot \mathbf{b}(\lambda)) = -q(\mathbf{l} \cdot \mathbf{m}) \quad \forall |\mathbf{l}| \leq 1, \forall |\mathbf{m}| \leq 1. \quad (25)$$

Thus, we arrive at

$$\int d\lambda p(\lambda) a_i(\lambda) b_j(\lambda) = -q\delta_{ij} \quad i, j \in \{x, y, z\}. \quad (26)$$

Now, we obtain the conditions, Eqs. (11), (12), (21), (22), and (26), which $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$ have to satisfy. Thus, these equations are necessary conditions for $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$. However, at the same time, they are sufficient conditions for $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$. In fact, from Eqs. (11), (12), (21), (22), and (26), we can always regenerate $W(q)$ defined in Eqs. (1) and (6). For example, using Eqs. (10), (11), (21), (22), and (26), we can calculate the matrix element $\langle 00|W(q)|00\rangle$ as follows:

$$\langle 00|W(q)|00\rangle = \frac{1}{4} \int d\lambda p(\lambda)(1 + a_z(\lambda))(1 + b_z(\lambda)) = \frac{1-q}{4}. \quad (27)$$

This result coincides with Eq. (6). We can obtain the similar results about the other matrix elements of $W(q)$.

If we define $p(\lambda)$, $\mathbf{a}(\lambda)$, and $\mathbf{b}(\lambda)$ as described below, they satisfy all of the necessary and sufficient conditions, Eqs. (11), (12), (21), (22), and (26). First, we define the variable λ as $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Second, we define the normalized probability distribution as

$$p(\theta, \phi) = \frac{1}{4\pi}. \quad (28)$$

Then we obtain

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta p(\theta, \phi) = 1, \quad (29)$$

and Eq. (11) is satisfied. Here, we pay attention to the fact that the volume element for the integral is given by $(d\theta d\phi \sin \theta)$ in Eq. (29).

Third, we define $\mathbf{a}(\theta, \phi)$ and $\mathbf{b}(\theta, \phi)$ as follows:

$$a_i(\theta, \phi) = \sqrt{3q} f_i(\theta, \phi), \quad b_i(\theta, \phi) = -\sqrt{3q} f_i(\theta, \phi), \quad (30)$$

where

$$\begin{cases} f_x(\theta, \phi) &= \sin \theta \cos \phi, \\ f_y(\theta, \phi) &= \sin \theta \sin \phi, \\ f_z(\theta, \phi) &= \cos \theta. \end{cases} \quad (31)$$

The functions f_x , f_y , and f_z satisfy the following relations:

$$\frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta f_i(\theta, \phi) = 0 \quad i \in \{x, y, z\}, \quad (32)$$

$$\frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta f_i(\theta, \phi) f_j(\theta, \phi) = \frac{1}{3} \delta_{ij} \quad i, j \in \{x, y, z\}. \quad (33)$$

From the above relations, we can confirm that $\mathbf{a}(\theta, \phi)$ and $\mathbf{b}(\theta, \phi)$ satisfy Eqs. (21), (22), and (26).

Here, let us calculate norms of \mathbf{a} and \mathbf{b} from Eqs. (30) and (31),

$$|\mathbf{a}| = |\mathbf{b}| = \sqrt{3q}. \quad (34)$$

Remembering Eq. (12), we can obtain a condition $0 \leq q \leq 1/3$ from Eq. (34). This implies that the explicit convex combination of product states of $W(q)$ given in this section is right only for $0 \leq q \leq 1/3$. This fact coincides with the condition for separability of $W(q)$. From this observation, we understand that the critical point $q = 1/3$ comes from positivity of local density operators, $\rho_A(\lambda)$ and $\rho_B(\lambda)$.

4 Discussions

In this paper, we write down the separable Werner state as a convex combination of product states explicitly, so that we construct a hidden variable model for the separable Werner state. Our convex combination for the separable Werner state is different from the convex combination obtained by Wootters' method.

In our decomposition, as shown in Eq. (30), $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ always point to the opposite directions with each other, namely, $\mathbf{a}(\lambda) = -\mathbf{b}(\lambda) \forall \lambda$. We cannot find any physical or geometrical meaning of this relation. We are not sure whether or not there exists a decomposed form that satisfies $\mathbf{a}(\lambda) \neq -\mathbf{b}(\lambda)$ for some λ .

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A The decomposed form obtained by Wootters' method for the separable Werner state

According to Ref. [11] written by Wootters, we can obtain the following convex combination of product states for the separable Werner state defined in Eq. (1) with $0 \leq q \leq 1/3$:

$$W(q) = \sum_{i=1}^4 |z_i\rangle\langle z_i|, \quad (35)$$

where

$$\begin{aligned} |z_1\rangle &= (1/2)(e^{i\theta_1}|x_1\rangle + e^{i\theta_2}|x_2\rangle + e^{i\theta_3}|x_3\rangle + e^{i\theta_4}|x_4\rangle), \\ |z_2\rangle &= (1/2)(e^{i\theta_1}|x_1\rangle + e^{i\theta_2}|x_2\rangle - e^{i\theta_3}|x_3\rangle - e^{i\theta_4}|x_4\rangle), \\ |z_3\rangle &= (1/2)(e^{i\theta_1}|x_1\rangle - e^{i\theta_2}|x_2\rangle + e^{i\theta_3}|x_3\rangle - e^{i\theta_4}|x_4\rangle), \\ |z_4\rangle &= (1/2)(e^{i\theta_1}|x_1\rangle - e^{i\theta_2}|x_2\rangle - e^{i\theta_3}|x_3\rangle + e^{i\theta_4}|x_4\rangle), \end{aligned} \quad (36)$$

$$\begin{aligned} |x_1\rangle &= -i\frac{\sqrt{1+3q}}{2}|\Psi^-\rangle, \\ |x_2\rangle &= \frac{\sqrt{1-q}}{2}|\Psi^+\rangle, \\ |x_3\rangle &= \frac{\sqrt{1-q}}{2}|\Phi^-\rangle, \\ |x_4\rangle &= -i\frac{\sqrt{1-q}}{2}|\Phi^+\rangle, \end{aligned} \quad (37)$$

$$\begin{aligned} |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B), \\ |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B), \end{aligned} \quad (38)$$

and

$$e^{-2i\theta_1}(1+3q) + (e^{-2i\theta_2} + e^{-2i\theta_3} + e^{-2i\theta_4})(1-q) = 0. \quad (39)$$

Equation (39) does not determine θ_1 , θ_2 , θ_3 , and θ_4 uniquely. For example, we have a special solution, $\theta_1 = 0$, $\theta_2 = \pi/2$,

$$\cos \theta_3 = \sqrt{\frac{1-3q}{2(1-q)}}, \quad \sin \theta_3 = \sqrt{\frac{1+q}{2(1-q)}}, \quad (40)$$

and

$$\cos \theta_4 = -\sqrt{\frac{1-3q}{2(1-q)}}, \quad \sin \theta_4 = \sqrt{\frac{1+q}{2(1-q)}}. \quad (41)$$

By substituting the above special solution into Eqs. (36), (37), and (38), we can confirm that $|z_1\rangle$, $|z_2\rangle$, $|z_3\rangle$, and $|z_4\rangle$ are product states.

References

- [1] A. Einstein, B. Podolsky, and N. Rosen, ‘Can quantum-mechanical description of physical reality be considered complete’, *Phys. Rev.* **47**, 777–780 (1935).
- [2] J.S. Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge, 1987), Chaps. 1 and 2.
- [3] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, ‘Proposed experiment to test local hidden-variable theories’, *Phys. Rev. Lett.* **23**, 880–884 (1969).
- [4] R.F. Werner, ‘Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model’, *Phys. Rev. A* **40**, 4277–4281 (1989).
- [5] S. Popescu, ‘Bell’s inequality and density matrices: Revealing “hidden” nonlocality’, *Phys. Rev. Lett.* **74**, 2619–2622 (1995).
- [6] A. Peres, ‘Separability criterion for density matrices’, *Phys. Rev. Lett.* **77**, 1413–1415 (1996).
- [7] M. Horodecki, P. Horodecki, and R. Horodecki, ‘Separability of mixed states: necessary and sufficient conditions’, *Phys. Lett. A* **223**, 1–8 (1996).
- [8] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, ‘Mixed-state entanglement and quantum error correction’, *Phys. Rev. A* **54**, 3824–3851 (1996).
- [9] M. Murao, M.B. Plenio, S. Popescu, V. Vedral, and P.L. Knight, ‘Multiparticle entanglement purification protocols’, *Phys. Rev. A* **57**, R4075–R4078 (1998).
- [10] T. Hiroshima and S. Ishizaka, ‘Local and nonlocal properties of Werner states’, *Phys. Rev. A* **62**, 044302 (2000).
- [11] W.K. Wootters, ‘Entanglement of formation of an arbitrary state of two qubits’, *Phys. Rev. Lett.* **80**, 2245–2248 (1998).